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$$\therefore OH=\frac{b^2-2a^2}{2b}, \quad \therefore BH=\frac{a}{b}\sqrt{b^2-a^2}.$$

A'=area of segment BLC+area of segment BAC,

$$=\frac{b^2}{4}\left\{\sin^{-1}\left(\frac{2a}{b^2}\sqrt{b^2-a^2}\right)-\frac{2a(b^2-2a^2)\sqrt{b^2-a^2}}{b^4}\right\}$$

$$+a^2\left\{\sin^{-1}\frac{\sqrt{b^2-a^2}}{b}-\frac{a\sqrt{b^2-a^2}}{b^2}\right\},$$

$$=\tfrac{1}{4}b^2\cos^{-1}\left(\frac{b^2-2a^2}{b^2}\right)+a^2\cos^{-1}\frac{a}{b}-\frac{a}{2}\sqrt{b^2-a^2}.$$

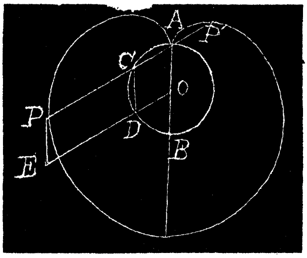
Also solved by *J. SCHEFFER* and *A. H. HOLMES*.

56. Proposed by B. F. BURLESON, Oneida Castle, New York.

Find (1) the length s of the closed curve of the cardioid; (2) its area A; (3) if made to revolve about its axis $2a$, find the maximum longitudinal circumference C of the solid generated; (4) find the surface K of the same; (5) its volume V; (6) the distance x_0 of the center of gravity of the solid from the origin O; and (7) the distance g_0 of the center of gravity of the plane curve from the origin O.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland, and the PROPOSER.

Let $AB=a$ be the diameter of a circle. From A draw any chord AC. Make CP and $CP'=b$, then will the locus of P or P' be the Limaçon. If $AP=r$, $\angle PAB=\theta$, we find at once the polar equation of the Limaçon to be $r=a\cos\theta+b$. If $b>a$, the curve consists of but one loop; if $b<a$, it has two loops, and if $b=a$ the curve becomes the Cardioid, the polar equation of which is $r=a(1+\cos\theta)$. It can easily be shown that the cardioid is an epicycloid, the generating circle of which is equal to the fixed one; also, drawing through the center O of the circle a line parallel to AP cutting the circumference of the circle at D, and drawing through P a line parallel to CD, this line is a tangent to the cardioid at P. The different problems proposed are best solved by means of the polar equation of the curve.



(1). The length $s=2\int_0^{\pi} d\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2},=2a\int_0^{\pi}\cos\tfrac{1}{2}\theta d\theta,=8a.$

(2). The area $A=\int_0^{\pi} r^2 d\theta,=a^2\int_0^{\pi}(1+\cos\theta)^2 d\theta,=\tfrac{3}{2}\pi a^2.$

(3). To find the maximum ordinate, $r\sin\theta = a(\sin\theta + \frac{1}{2}\sin 2\theta$ is to be a maximum. By differentiation we find $\theta = 60^\circ$. $\therefore$ maximum ordinate$=\frac{3}{4}a\sqrt{3}$, and circumference $C = \frac{3}{2}\pi a\sqrt{3}$.

(4). Surface $K = 2\pi\int_0^\pi r\sin\theta d\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)}$,

$$= 4\pi a^2\int_0^\pi (1+\cos\theta)\sin\theta\cos\tfrac{1}{2}\theta d\theta, = -16\pi a^2\int_0^\pi \cos^4\tfrac{1}{2}\theta d\cos\tfrac{1}{2}\theta, = \frac{32\pi a^2}{5}.$$

For the distance x_0 of the center of gravity of this surface we have

$$Kx_0 = 2\pi\int_0^\pi r^2\sin\theta\cos\theta(ds/d\theta).d\theta, = 4\pi a^3\int_0^\pi (1+\cos\theta)^2\sin\theta\cos\theta.\cos\tfrac{1}{2}\theta d\theta,$$

$$= -64\pi a^3\,[2\int_0^\pi \cos^8\tfrac{1}{2}\theta d\cos\tfrac{1}{2}\theta.\int_0^\pi \cos^6\tfrac{1}{2}\theta d\cos\tfrac{1}{2}\theta], = \frac{320\pi a^3}{63};$$

$$\therefore x_0 = \frac{320\pi a^3}{63} \div \frac{32\pi a^2}{5} = \frac{50}{63}a.$$

(5). Volume, $V = 2\pi\int_0^\pi\int_0^{a(1+\cos\theta)} rdrd\theta.r\sin\theta, = \frac{2\pi a^3}{3}\int_0^\pi (1+\cos\theta)^3\sin\theta d\theta$

$$= -\frac{2\pi a^3}{3}\int_0^\pi (1+\cos\theta)^3 d(1+\cos\theta), = \frac{8\pi a^3}{3}.$$

(6). The distance x_0 of this volume from the origin we find from

$$Vx_0 = 2\pi\int_0^\pi\int_0^{a(1-\cos\theta)} r^2 drd\theta\sin\theta r\sin\theta, = \frac{\pi a^4}{2}\int_0^\pi (1+\cos\theta)^4\sin^2\theta d\theta,$$

$$= 32\pi a^4\int_0^\pi \cos^{10}\tfrac{1}{2}\theta\sin^2\tfrac{1}{2}\theta d\theta, = \frac{64\pi a^4\Gamma(\frac{3}{2})\Gamma(\frac{11}{2})}{2\Gamma(7)}, = \frac{21}{22}\pi^2 a^4;$$

$$\therefore x_0 = \frac{21\pi^2 a^4}{32} \div \frac{8\pi a^3}{3} = \frac{63}{256}\pi a.$$

(7). The distance x_0 of the center of gravity of the arc of the curve from the origin is found by

$$sx_0 = 2\int_0^\pi r\sin\theta.2a\cos\tfrac{1}{2}\theta.d\theta, = 16a^2\int_0^\pi \cos^4\tfrac{1}{2}\theta.\sin\tfrac{1}{2}\theta.d\theta,$$

$$= -32a^2\int_0^\pi \cos^4\tfrac{1}{2}\theta d(\cos\tfrac{1}{2}\theta), = \frac{32a^2}{5}, \quad \therefore x_0 = \frac{32a^2}{5} \div 8a = \frac{4}{5}a;$$

and the distance x'_0 of the area of the curve from the origin is found by

$$Ax'_0=\tfrac{2}{3}\int_0^{\pi} r^3\cos\theta d\theta,=\tfrac{2}{3}a^3\int_0^{\pi}(1+\cos\theta)^3\cos\theta d\theta,=\tfrac{16}{3}a^2\int_0^{\pi}(2\cos^8\tfrac{1}{2}\theta-\cos^6\tfrac{1}{2}\theta)d\theta,$$

$$=\tfrac{5}{4}\pi a^3. \qquad \therefore\ x'_0=\tfrac{5}{4}\pi a^3\div\frac{3\pi a^2}{2}=\tfrac{5}{6}a.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $r=a(1+\cos\theta)$, be the equation to the Cardioid.

(1). $s/2=2a\int_0^{\pi}\cos(\tfrac{1}{2}\theta)d\theta,=4a. \quad \therefore\ s=8a.$

(2). $A=2a^2\int_0^{2\pi}\cos^4(\tfrac{1}{2}\theta)d\theta,=\tfrac{1}{2}(3\pi a^2).$

(3). $C=2\pi\rho$, where $\rho=r\sin\theta,=a\sin\theta(1+\cos\theta)$. $d\rho=2a\cos^2\theta+a\cos\theta-a.$

$\therefore$ $\cos\theta=\tfrac{1}{2}$ or -1. $\therefore$ for a maximum $\theta=60^\circ$.

$\therefore$ $C=2\pi a(1+\cos\tfrac{1}{3}\pi)\sin\tfrac{1}{3}\pi,=\tfrac{1}{2}(3\sqrt{3}\pi a).$

(4). $K=8\pi a^2\int_0^{\pi}\cos^3(\tfrac{1}{2}\theta)\sin\theta d\theta,=\tfrac{1}{5}(32\pi a^2).$

(5). $$V=2\pi\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\sin\theta d\theta dr,=\frac{16\pi a^3}{3}\int_0^{\pi}\cos^6(\tfrac{1}{2}\theta)\sin\theta d\theta.$$

$\therefore$ $V=\tfrac{1}{3}(8\pi a^3)$.

(6). $$x_0=\frac{\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3\sin\theta\cos\theta d\theta dr}{\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\sin\theta d\theta dr},=\tfrac{3}{2}a\frac{\int_0^{\pi}\cos^8(\tfrac{1}{2}\theta)\cos\theta\sin\theta d\theta}{\int_0^{\pi}\cos^6(\tfrac{1}{2}\theta)\sin\theta d\theta}$$

$\therefore$ $x_0=\tfrac{4}{5}a$.

(7). $$y_0=\frac{\int_0^{2\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta d\theta dr}{\int_0^{2\pi}\int_0^{a(1+\cos\theta)} r d\theta dr},=\tfrac{4}{3}a\frac{\int_0^{2\pi}\cos^6(\tfrac{1}{2}\theta)\cos\theta d\theta}{\int_0^{2\pi}\cos^4(\tfrac{1}{2}\theta)d\theta}$$

$\therefore$ $y_0=\tfrac{5}{6}a$.

Also solved by *C. W. M. BLACK.*